Control Theory and Algebraic Geometry [1]

or

How some questions of algebraic geometry appear in the theory of control of systems described by partial differential equations

Shiva Shankar

Chennai Mathematical Institute,
Plot H1, SIPCOT, IT Park, Padur P.O.,
Siruseri, Chennai (Madras) - 603103,
India.

Abstract

In this talk I explain the *behavioural* generalisation of Kalman's fundamental notion of *state space controllability* [2] due to Willems [13], and beyond to systems described by PDE [6]. It turns out that controllability is now identical to the notion of a *potential* in Physics or *vanishing homology* in Mathematics.

This development relies and builds on classical work of Hörmander, Malgrange, Palamodov and others in PDE and Several Complex Variables. It also requires setting up PDE equivalents of fundamental notions in Algebraic Geometry - the radical of an ideal, the Hilbert Nullstellensatz, a complete variety, the elimination problem.

This subject thus lies in the intersection of many important streams, from Physics, Mathematics and Engineering [10].

1. Introduction

I am grateful for this opportunity to speak here.

Given the interests of this audience, I would like to immediately announce the idea which the rest of my talk will hopefully justify - and it is this:

If $\mathcal{A} = \mathbb{C}[X_1, \ldots, X_n]$, and $p(X)$ is in $\mathcal{A}$, then x in $\mathbb{C}^n$ is a zero of $p(X)$ if $p(x) = 0$. The variety $\mathcal{V}(p)$ is the set of zeros of p. If $\mathfrak{i}$ is an ideal of $\mathcal{A}$, then $\mathcal{V}(\mathfrak{i})$ is the set of common zeros of all the polynomials in $\mathfrak{i}$.

If instead $\mathcal{A} = \mathbb{C}\,[\partial_1, \ldots, \partial_n]$, and $p(\partial)$ is in $\mathcal{A}$, then a distribution f in $\mathcal{D}'$ is a zero of $p(\partial)$ if $p(\partial)f = 0$. More generally, instead of $\mathcal{D}'$, one can consider zeros in some $\mathcal{A}$-submodule $\mathcal{F}$ of $\mathcal{D}'$. Even more generally, if $p(\partial) = (p_1(\partial), \ldots, p_k(\partial))$ is in $\mathcal{A}^k$, then $f = (f_1, \ldots, f_k)$ in $\mathcal{F}^k$ is a zero of $p(\partial)$ if $p(\partial)f = \sum p_i(\partial) f_i = 0$. Let $\mathcal{B}_\mathcal{F}(p)$ denote the set of zeros of p in $\mathcal{F}^k$. If $\mathcal{P}$ is a submodule of $\mathcal{A}^k$, then $\mathcal{B}_\mathcal{F}(\mathcal{P})$ is the set of common zeros of all the elements in $\mathcal{P}$.

Given this analogy, one can ask questions about these zeros in $\mathcal{F}^k$ corresponding to questions in algebraic geometry, such as, what is the analogue of a radical ideal, of the Hilbert Nullstellensatz, etc?

I hope to illustrate in the rest of my talk that these are not frivolous questions, but that they suggest themselves in the theory of control of dynamical systems.

[1] Talk presented at the 21st conference of the Ramanujan Mathematical Society, Central University, Hyderabad, July, 2006.

2. Controllability

I would like to right away define the notion of controllability of a dynamical system.

This notion was first introduced by R.E.Kalman [2] in the 1960s for dynamical systems that admit a phase or state space model. It was quickly recognised to be a fundamental notion, and it was on this that the superstructure of post-war control theory was built. I shall not speak of phase space models here, but shall speak about a more general class of models which describe a much larger class of phenomena. To explain the notion of controllability in this context, I must first say a few words about modelling, and the class of models that a theory focusses its attention on.

A model is a picture of reality, and the closer it is to reality, the better the picture it will be, and the more effective a theory will be the one that describes this model. A model seeks to represent a certain phenomenon or system whose attributes are certain qualities that are changing with space, time etc. The closest we can get to this reality, this phenomenon, is to take all possible variations of the attributes of the phenomenon, itself, as the model. This collection, considered all together in our minds, is, in engineering parlance, the *behaviour* of the phenomenon or the system.

A priori, any evolution of these attributes could perhaps have occured, but the *laws* of the system prescribe those evolutions that actually do. I shall consider behaviours described by local laws, i.e. laws expressed by differential equations. (Here by local I mean that the variation of the attributes of the phenomenon at a point depends on the values of the attributes in arbitrarily small neighbourhoods of the point - and not on points far away.)

This is a familiar situation in Physics, Mathematics and Engineering. For instance

1. Planetary motion. A priori, the earth could perhaps have traversed any trajectory around the sun, but Kepler's laws restrict it to travel an elliptic orbit, sweeping equal areas in equal times, and such that the square of the period of revolution is proportional to the cube of the major semi-axis.

Kepler's laws are of course local, namely Newton's equations of motion.

2. Magnetic fields. A priori, any function B could perhaps have been a magnetic field, but in fact must satisfy the law that there do not exist magnetic monopoles in the universe. This law is also local:

$$\mathsf{div}\, B = 0$$

3. Complex analysis! Complex analysis studies those functions which admit a convergent power series about each point. This is the law that defines the subject.

This law is again local, viz. the Cauchy-Riemann equations.

I now further confine myself to behaviours which are not only described by local laws, but are also linear and shift invariant. Such behaviours arise as kernels of maps given by systems of constant coefficient differential operators.

So suppose that the attributes of a system are described by some k-tuple of smooth functions, $f = (f_1, \ldots, f_k)$, $f_i : \mathbb{R}^n \to \mathbb{C}$. Then a law the system obeys is a partial differential operator $p(\partial) = (p_1(\partial), \ldots, p_k(\partial))$ such that $p(\partial)f = \sum p_i(\partial) f_i = 0$. In other words a law is an operator $p(\partial) : (\mathcal{C}^\infty)^k \to \mathcal{C}^\infty$, and to say that f obeys this law is to say that $p(\partial)f = 0$, or that f is a zero of $p(\partial)$. The phenomenon itself is

$$\mathcal{B} = \bigcap_{\text{all laws } p} \{f | p(\partial) f = 0\}$$

the common zeros of all the laws governing the phenomena.

Shift-invariance implies that the entries of $p(\partial)$ are from the ring $\mathcal{A} = \mathbb{C}[\partial_1, \ldots, \partial_n]$ of constant coefficient partial differential operators on $\mathbb{R}^n$. Linearity implies that the totality of laws the system obeys is the submodule $\mathcal{P}$ of $\mathcal{A}^k$ generated by all the laws $p(\partial)$. Thus we assume that the sum of two laws is a law, and that multiplying a law by a partial differential operator is also a law. As $\mathcal{A}$ is Noetherian, all these laws are generated by a finite number of laws. If they be l in number, then writing these l laws as rows of a matrix gives an operator

$$P(\partial) : (\mathcal{C}^\infty)^k \longrightarrow (\mathcal{C}^\infty)^l$$

whose kernel is precisely the behaviour of the system.

There is nothing very special about the $\mathcal{A}$-module $\mathcal{C}^\infty$, and I shall more generally consider the $\mathcal{A}$-modules $\mathcal{D}'$, $\mathcal{C}^\infty$, the space $\mathcal{S}'$ of temperate distributions, the Schwartz space $\mathcal{S}$, and the spaces $\mathcal{E}'$ and $\mathcal{D}$ of compactly supported distributions and smooth functions - in short the *classical* spaces.

Notation: Given an $\mathcal{A}$-submodule $\mathcal{F}$ of $\mathcal{D}'$, denote by $\mathsf{Ker}_{\mathcal{F}}\,\mathcal{P}$ (or, as in the introduction, by $\mathcal{B}_{\mathcal{F}}(\mathcal{P})$) the common zeros in $\mathcal{F}$ of all the elements in the submodule $\mathcal{P}$. I shall also call such a behaviour a *differential kernel in* $\mathcal{F}$.

Definition [6]: Let $\mathcal{B}$ be a behaviour - in this talk a differential kernel. Then $\mathcal{B}$ is said to be *controllable* if for any two subsets U and V of $\mathbb{R}^n$ whose closures do not intersect, and f and g any two elements in $\mathcal{B}$, there is an h in $\mathcal{B}$ such that h equals f on U and g on V.

Thus controllability asserts the possibility of solving a *patching problem.*

As I remarked earlier, Kalman had introduced a notion of controllability for phase space systems described by ordinary differential equations, i.e. where the attributes of the system are functions only of time. It turns out that in this 1-dimensional situation, the above patching definition, when restricted to these phase space systems, is precisely the Kalman definition [13].

At this point I must also make another remark. This notion of controllability is a very engineering notion. By this I mean the following - systems, i.e. differential kernels, that appear in Mathematics are, generally speaking, over-determined. Indeed the theory of $\mathcal{D}$-modules concentrates attention on holonomic or maximally over-determined systems. Here the characteristic variety of the system is of dimension 0, the space of solutions is a finite dimensional vector space, and the questions of importance are arithmetic ones.

On the other hand, the systems of interest in engineering, i.e. those manufactured by humans, are under-determined. This must be so, for engineering systems must be manouverable, even 'controllable'. Thus one should, ideally speaking, be able to move from one trajectory to another. This inbuilt freedom is not a property of holonomic systems, but of under-determined ones. And I shall explain that controllability is a feature of maximally under-determined systems, that is those whose characteristic variety is all of $\mathbb{C}^n$. The systems we manufacture are thus at the very other end from being holonomic.

3. Elimination

In practice, while developing a model for a system or phenomenon, one will need to introduce variables other than those that describe the attributes of interest. For instance in describing the variations of electric and magnetic fields, Maxwell's equations require the introduction of other variables such as current and charge distribution. This will lead to models of the form

$$\mathcal{B} = \{f \in \mathcal{F}^k | P(\partial)f = M(\partial)g\}$$

The pairs (f, g) above is of course a kernel (that of the operator $(P(\partial), -M(\partial))$) and $\mathcal{B}$ a projection of this kernel. The question now is - is a projection of a kernel also a kernel? Or rather

Is a projection of the zeros of a differential operator also the zeros of some differential operator?

This is the elimination problem for PDE, and the answer depends on the space $\mathcal{F}$ in which the zeros are located.

Example: Let $\mathcal{A} = \mathbb{C}[\frac{d}{dt}]$, and let $\pi_2 : \mathcal{D}^2 \to \mathcal{D}$ be the projection onto the second factor. Let $\mathcal{P}$ be the (cyclic) submodule of $\mathcal{A}^2$ generated by $(\frac{d}{dt}, -1)$. Then $\mathsf{Ker}_{\mathcal{D}}\,\mathcal{P} = \{(f, \frac{df}{dt}) | f \in \mathcal{D}\}$, but $\pi_2\,\mathsf{Ker}_{\mathcal{D}}\,\mathcal{P} = \{\frac{df}{dt} | f \in \mathcal{D}\}$ is not a differential kernel in $\mathcal{D}$. □

Definition [11]: Let $\mathcal{F}$ be an $\mathcal{A}$-submodule of $\mathcal{D}'$. Suppose that all the projections of every differential kernel in $\mathcal{F}$ are also differential kernels in $\mathcal{F}$. Then $\mathcal{F}$ is said to be geometrically complete.

Thus $\mathcal{D}$ is not geometrically complete, nor are $\mathcal{E}'$ and $\mathcal{S}$. On the other hand $\mathcal{D}'$, $\mathcal{C}^\infty$ and $\mathcal{S}'$ are geometrically complete. That elimination is always possible in these spaces is a consequence of an old (1960s) and famous result - The Fundamental Principle of Malgrange-Palamodov.

This result states that every image $(\mathcal{D}')^k \xrightarrow{P(\partial)} (\mathcal{D}')^l$ is also a kernel, in fact the kernel of $(\mathcal{D}')^l \xrightarrow{Q(\partial)} (\mathcal{D}')^m$, where the m rows of $Q(\partial)$ generate all the relations between the rows of $P(\partial)$. Thus

$$(\mathcal{D}')^k \xrightarrow{P(\partial)} (\mathcal{D}')^l \xrightarrow{Q(\partial)} (\mathcal{D}')^m$$

is exact.

The Fundamental Principle answers the solvability question for systems of PDE: given g in $(\mathcal{D}')^l$, is there an f in $(\mathcal{D}')^k$ such that $P(\partial)f = g$? The principle asserts - yes, if and only if $Q(\partial)g = 0$ [1,5].

Example: Consider the curl operator

$$\mathsf{curl} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{pmatrix}$$

The relations between its rows is generated by $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$. Thus $\mathsf{curl}\, f = g$ is solvable for a given g in $(\mathcal{D}')^3$ if and only if $\mathsf{div}\, g = 0$. This is just the exactness of

$$(\mathcal{D}')^3 \xrightarrow{\mathsf{curl}} (\mathcal{D}')^3 \xrightarrow{\mathsf{div}} \mathcal{D}' \tag{1}$$

Similarly the relations between the rows of the gradient operator $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})^t$ are generated by the rows of curl. Then again by the Fundamental Principle $\mathsf{grad}\, f = g$ is solvable for a given g in $(\mathcal{D}')^3$ if and only if $\mathsf{curl}\, g = 0$. It similarly follows that $\mathsf{div}\, f = g$ can be solved for every g, so that

$$\mathcal{D}' \xrightarrow{\mathsf{grad}} (\mathcal{D}')^3 \xrightarrow{\mathsf{curl}} (\mathcal{D}')^3 \xrightarrow{\mathsf{div}} \mathcal{D}' \to 0$$

is exact. □

I wish to emphasise the role played by the space $\mathcal{F}$ where the solutions are located (lest it be thought that this is a problem in algebra). For instance it turns out that not only is (1) exact, but that also its restriction to $\mathcal{D}$

$$\mathcal{D}^3 \xrightarrow{\mathsf{curl}} \mathcal{D}^3 \xrightarrow{\mathsf{div}} \mathcal{D}$$

is exact. Now, as the columns $c_1(\partial), c_2(\partial), c_3(\partial)$ of curl are $\mathcal{A}$-dependent, i.e. as $\frac{\partial}{\partial x}c_1(\partial) + \frac{\partial}{\partial y}c_2(\partial) + \frac{\partial}{\partial z}c_3(\partial) = 0$, the $\mathcal{D}'$ image of curl is also the $\mathcal{D}'$ image of the operator $\mathsf{crl} = (c_1(\partial), c_2(\partial))$ given by the first two columns of curl. For suppose g is in $\mathcal{D}'$. Let f in $\mathcal{D}'$ be such that $\frac{\partial}{\partial z}f = g$ (by the Fundamental Principle there is such an f - indeed if $p(\partial)$ is nonzero, then there is always an f such that $p(\partial)f = g$ - which is to say that $\mathcal{D}'$ is a divisible $\mathcal{A}$-module). Hence $c_3(\partial)g = c_3(\partial)\frac{\partial}{\partial z}f = -(\frac{\partial}{\partial x}c_1(\partial) + \frac{\partial}{\partial y}c_2(\partial))f = c_1(\partial)(-\frac{\partial}{\partial x}f) + c_2(\partial)(-\frac{\partial}{\partial y}f)$, which is in the image of the operator crl.

In other words

$$(\mathcal{D}')^2 \xrightarrow{\mathsf{crl}} (\mathcal{D}')^3 \xrightarrow{\mathsf{div}} \mathcal{D}'$$

is also exact. On the other hand its restriction to $\mathcal{D}$

$$\mathcal{D}^2 \xrightarrow{\mathsf{crl}} \mathcal{D}^3 \xrightarrow{\mathsf{div}} \mathcal{D}$$

is *not* exact [11].

In the above I used the Fundamental Principle in the simplest situation, that for a single PDE. It asserted the surjectivity of $\mathcal{D}' \xrightarrow{p(\partial)} \mathcal{D}'$, viz. that $\mathcal{D}'$ is a divisible $\mathcal{A}$-module. So is $\mathcal{C}^\infty$ - this is the basic existence theorem of Ehrenpreis-Malgrange. The space $\mathcal{S}'$ of temperate distributions is also a divisible $\mathcal{A}$-module - this is a famous result of Hörmander-Łojasiewicz, and in particular it implies the existence of fundamental solutions that are temperate.

But the Fundamental Principle says much more: the sequence $\mathcal{F}^k \xrightarrow{P(\partial)} \mathcal{F}^l \xrightarrow{Q(\partial)} \mathcal{F}^m$ is just the sequence $\mathsf{Hom}_\mathcal{A}(\mathcal{A}^k, \mathcal{F}) \xrightarrow{P(\partial)} \mathsf{Hom}_\mathcal{A}(\mathcal{A}^l, \mathcal{F}) \xrightarrow{Q(\partial)} \mathsf{Hom}_\mathcal{A}(\mathcal{A}^m, \mathcal{F})$, and to say this is exact when $\mathcal{A}^m \xrightarrow{Q^t(\partial)} \mathcal{A}^l \xrightarrow{P^t(\partial)} \mathcal{A}^k$ is exact is to say that $\mathcal{F}$ is an injective module.

Indeed, $\mathcal{D}'$ and $\mathcal{C}^\infty$ are injective co-generators, whereas $\mathcal{S}'$ is injective though not a co-generator [3,5,11].

On the other hand $\mathcal{S}$, $\mathcal{E}'$ and $\mathcal{D}$ are not injective, not even divisible $\mathcal{A}$-modules, and we have seen that elimination is not always possible. The problem now is to locate the obstruction to elimination in some $\mathcal{A}$-module. This is indeed possible because $\mathsf{Ker}_{\mathcal{F}}\mathcal{P} \simeq \mathsf{Hom}_{\mathcal{A}}(\mathcal{A}^k/\mathcal{P}, \mathcal{F})$, and it turns out that the above obstruction can be located in an $\mathsf{Ext}^1_{\mathcal{A}}$ module [11].

4. The category of differential kernels

I have already suggested that there is a category of differential kernels somewhat like the category of affine varieties, and that some of the questions about the former are analogues of questions about the latter. A most basic fact about the category of affine varietes is that there is a duality between this category and the category of finitely generated, nilpotent-free $\mathbb{C}$-algebras. What then is the corresponding statement for differential kernels?

So let $\mathcal{R} = \mathsf{End}_{\mathcal{A}}(\mathcal{F})$ be the $\mathcal{A}$-algebra of all $\mathcal{A}$-linear endomorphisms of $\mathcal{F}$. Then $\mathcal{F}$ is an $\mathcal{R}$-module via $rf = r(f)$ for f in $\mathcal{F}$ (in fact $\mathcal{F}$ is an $\mathcal{A}$-$\mathcal{R}$ bimodule). If $\mathcal{M}$ is an $\mathcal{A}$-module, the $\mathcal{A}$-module $\mathsf{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{F})$ is an $\mathcal{A}$-$\mathcal{R}$ bimodule via $r\phi = r \circ \phi$ (composition).

There is now (as in the category of affine varieties) a contravariant functor

$$\begin{array}{rcl} \mathsf{Hom}_{\mathcal{A}}(\ \ , \mathcal{F}) : \mathcal{A} - \mathsf{Mod} & \longrightarrow & \mathcal{R} - \mathsf{Mod} \\ \mathcal{M} & \mapsto & \mathsf{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{F}) \\ \mathcal{M} \xrightarrow{\phi} \mathcal{N} & \mapsto & \mathsf{Hom}_{\mathcal{A}}(\mathcal{N}, \mathcal{F}) \xrightarrow{\circ\phi} \mathsf{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{F}) \end{array}$$

The canonical $\mathcal{A}$-isomorphism $\mathsf{Hom}_{\mathcal{A}}(\mathcal{A}^k, \mathcal{F}) \simeq \mathcal{F}^k$ is also an $\mathcal{R}$-isomorphism, hence

$$\begin{array}{rcl} \mathsf{Hom}_{\mathcal{A}}(\mathcal{A}^k/\mathcal{P}, \mathcal{F}) & \simeq & \mathsf{Ker}_{\mathcal{F}}\mathcal{P} \\ \phi & \mapsto & (\phi(\bar{e}_1), \ldots, \phi(\bar{e}_k)) \end{array}$$

is also an $\mathcal{A}$-$\mathcal{R}$ isomorphism (the $\bar{e}_i$ s are the images of the standard basis of $\mathcal{A}^k$ in $\mathcal{A}^k/\mathcal{P}$).

Thus, there is a contravariant functor between differential kernels in $\mathcal{F}$ and the full subcategory of finitely generated $\mathcal{A}$-modules.

By a fundamental result of Oberst [4] this functor establishes a *categorical duality* when $\mathcal{F}$ is $\mathcal{D}'$ or $\mathcal{C}^\infty$ (more precisely he shows that $\mathcal{D}'$ and $\mathcal{C}^\infty$ are *large* injective cogenerators). This raises the

Question: Which is the sub-family of finitely generated $\mathcal{A}$-modules that is in duality (via the above functor) with differential kernels in the other classical spaces? Or rather

What is the equivalent of the Hilbert Nullstellensatz for systems of PDE?

Some other questions about differential kernels corresponding to questions about affine varieties -

Are the intersection and sum of two differential kernels also differential kernels?

While the intersection of two differential kernels is always one, the answer for the sum depends on $\mathcal{F}$ [9,12].

Example: In a classical space $\mathcal{F}$ it turns out that if $\mathsf{Ker}_{\mathcal{F}}\mathcal{P}_1 + \mathsf{Ker}_{\mathcal{F}}\mathcal{P}_2$ were a differential kernel, then it must necessarily equal $\mathsf{Ker}_{\mathcal{F}}(\mathcal{P}_1 \cap \mathcal{P}_2)$.

Now let $\mathcal{A}$=$\mathbb{C}[\frac{d}{dt}]$ and let $\mathcal{F} = \mathcal{D}$. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be (cyclic) submodules of $\mathcal{A}^2$ generated by $(1, 0)$ and $(1, -\frac{d}{dt})$ respectively. Then $\mathcal{P}_1 \cap \mathcal{P}_2 = 0$, so that $\mathsf{Ker}_{\mathcal{D}}(\mathcal{P}_1 \cap \mathcal{P}_2) = \mathcal{D}^2$. On the other hand $\mathsf{Ker}_{\mathcal{D}}\,\mathcal{P}_1 = \{(0,f)|f \in \mathcal{D}\}$ and $\mathsf{Ker}_{\mathcal{D}}\,\mathcal{P}_2 = \{(\frac{dg}{dt}, g)|g \in \mathcal{D}\}$. Thus an element (u,v) in $\mathcal{D}^2$ is in $\mathsf{Ker}_{\mathcal{D}}\,\mathcal{P}_1 + \mathsf{Ker}_{\mathcal{D}}\,\mathcal{P}_2$ if and only if $u = \frac{dg}{dt}$, $v = f + g$, where f and g are arbitrary elements in $\mathcal{D}$. Let now u be any (nonzero) non-negative function in $\mathcal{D}$. Then $(u,0)$, which is in $\mathsf{Ker}_{\mathcal{D}}(\mathcal{P}_1 \cap \mathcal{P}_2)$, is however not in $\mathsf{Ker}_{\mathcal{D}}\,\mathcal{P}_1 + \mathsf{Ker}_{\mathcal{D}}\,\mathcal{P}_2$, as $g(t) = \int_{-\infty}^{t} dg = \int_{-\infty}^{t} u\,dt$ is not compactly supported. Thus this sum cannot be a differential kernel. □

Question: What is the obstruction to a sum of differential kernels being a differential kernel?
Answer: This obstruction can be located in $\mathsf{Ext}^1_{\mathcal{A}}(\mathcal{A}^k/(\mathcal{P}_1 + \mathcal{P}_2), \mathcal{F})$ [12].

5. Back to controllability

The Fundamental Principle (for the spaces $\mathcal{D}'$, $\mathcal{C}^\infty$, $\mathcal{S}'$) is the answer to the solvability question, and asserts that every image is a kernel.

What about the dual statement, viz. when is a kernel an image? Thus given $\mathcal{F}^k \stackrel{P(\partial)}{\longrightarrow} \mathcal{F}^l$, when is there an $\mathcal{F}^r \stackrel{R(\partial)}{\longrightarrow} \mathcal{F}^k$ such that

$$\mathcal{F}^r \stackrel{R(\partial)}{\longrightarrow} \mathcal{F}^k \stackrel{P(\partial)}{\longrightarrow} \mathcal{F}^l$$

is exact?

This is an important question in Physics, and such an $R(\partial)$ is called a potential. For instance the exactness of the sequence (1) translates to the statement that the set of magnetic fields (which is the kernel of the divergence operator) admits a 'vector potential'.

In the context of the subject matter of my talk, it turns out that the patching problem of the controllability question can be solved precisely when the kernel (specifying the behaviour) is an image.

Theorem[6,8]: A behaviour (in a classical space) given by the kernel of a differential operator is controllable if and only if it is an image. □

The question now is - when is $\mathsf{Ker}_{\mathcal{F}}\,\mathcal{P}$ an image? The answer of course depends on $\mathcal{F}$ (and $\mathcal{P}$). For instance, in $\mathcal{D}'$ and $\mathcal{C}^\infty$ the answer is - if and only if $\mathcal{A}^k/\mathcal{P}$ is torsion free. For such a system of PDE, its characteristic variety is all of $\mathbb{C}^n$, and $\mathsf{Ker}_{\mathcal{D}}\,\mathcal{P}$ is dense in $\mathsf{Ker}_{\mathcal{C}^\infty}\,\mathcal{P}$. Thus a controllable system is antipode to a holonomic one.

In the spaces $\mathcal{D}$, $\mathcal{E}'$ and $\mathcal{S}$, every kernel is an image. Indeed, $\mathcal{D}$ and $\mathcal{E}'$ are faithfully flat $\mathcal{A}$-modules, and $\mathcal{S}$ is a flat (though not faithfully flat) $\mathcal{A}$-module.

The general question here is thus - when is an image a kernel, and when is a kernel an image? The answers for the classical spaces are in terms of the associated primes of $\mathcal{A}^k/\mathcal{P}$ [11].

6. The Nullstellensatz for systems of PDE

It turns out that the answers to the above questions depend upon a PDE analogue of the Hilbert Nullstellensatz.

Let $\mathcal{P}$ be a submodule of $\mathcal{A}^k$, and $\mathcal{F}$ an $\mathcal{A}$-module. Let $\mathsf{MKer}_{\mathcal{F}}(\mathcal{P})$ be the submodule of $\mathcal{A}^k$ consisting of all those $p(\partial)$ whose kernel (in $\mathcal{F}$) contains $\mathsf{Ker}_{\mathcal{F}}\,\mathcal{P}$. Clearly this submodule contains $\mathcal{P}$, and in fact is the largest submodule of $\mathcal{A}^k$ whose kernel in $\mathcal{F}$ equals $\mathsf{Ker}_{\mathcal{F}}\,\mathcal{P}$. Call this submodule the *closure* of $\mathcal{P}$ in $\mathcal{F}$. Call $\mathcal{P}$ *closed* with respect to $\mathcal{F}$ if it equals its closure. This is completely analogous to the familiar Galois correspondence between ideals and varieties. The notion of closure here is the analogue of the radical of an ideal, and its calculation is the analogue of the Hilbert Nullstellensatz.

The calculation of this closure for the classical spaces is

Theorem[8,9]: (i) Every submodule $\mathcal{P}$ is closed with respect to $\mathcal{D}'$ or $\mathcal{C}^\infty$ (this is the Fundamental Principle).
(ii) Let $\mathcal{P} = \bigcap_{i=1}^t \mathcal{Q}_i$ be an irredundant primary decomposition of $\mathcal{P}$ in $\mathcal{A}^k$, where $\mathcal{Q}_i$ is $\mathfrak{p}_i$-primary. Suppose that the affine varieties in $\mathbb{C}^n$ of $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ contain purely imaginary points (i.e. intersect $\imath\mathbb{R}^n$) and that of $\mathfrak{p}_{s+1}, \ldots, \mathfrak{p}_t$ do not. Then the closure of $\mathcal{P}$ with respect to $\mathcal{S}'$ is $\bigcap_{i=1}^s \mathcal{Q}_i$, so that $\mathcal{P}$ is closed with respect to $\mathcal{S}'$ if and only if the variety of every associated prime of $\mathcal{A}^k/\mathcal{P}$ contains purely imaginary points.
(iii) Let $\pi : \mathcal{A}^k \to \mathcal{A}^k/\mathcal{P}$ be the canonical projection. Then the closure of $\mathcal{P}$ with respect to $\mathcal{S}$, $\mathcal{E}'$ or $\mathcal{D}$ is $\pi^{-1}(T(\mathcal{A}^k/\mathcal{P}))$, where $T(\mathcal{A}^k/\mathcal{P})$ is the submodule of torsion elements of $\mathcal{A}^k/\mathcal{P}$. Thus $\mathcal{P}$ is closed with respect to any of these spaces if and only if $\mathcal{A}^k/\mathcal{P}$ is torsion free (or equivalently if and only if $\mathcal{P}$ is 0-primary). □

Thus $\mathsf{Ker}_{\mathcal{D}'}\,\mathcal{P}$ is controllable if and only if $\mathcal{P}$ is closed with respect to $\mathcal{D}$.

There are other related questions which can be answered by these methods. For instance

Corollary[8]: For any submodule $\mathcal{P}$ of $\mathcal{A}^k$, $\mathsf{Ker}_{\mathcal{D}}\,\mathcal{P}$ is dense in $\mathsf{Ker}_{\mathcal{S}}\,\mathcal{P}$. Thus if $\mathsf{Ker}_{\mathcal{S}}\,\mathcal{P}$ is dense in $\mathsf{Ker}_{\mathcal{C}^\infty}\,\mathcal{P}$, then $\mathsf{Ker}_{\mathcal{D}}\,\mathcal{P}$ is itself dense in it. □

For these and other related matters, I refer to some papers below, as well as to the references in them.

Acknowledgements: I am grateful to the faculty and staff of the Department of Mathematics, Central University, Hyderabad, for their hospitality. I am also grateful to R.Sridharan for many discussions on many matters, especially on *The Brothers Karamazov*.

References

[1] L.Hörmander, *An Introduction to Complex Analysis in Several Variables*, 3rd edn, North Holland, Amsterdam, 1990.

[2] R.E.Kalman, On the general theory of control systems, *Proceedings of the 1st World Congress of the International Federation of Automatic Control*, 481-493, Moscow, 1960.

[3] B.Malgrange, Systèmes différentiels à coefficients constants, *Séminaire Bourbaki*, vol. 1962/ 63:246.01-246.11, 1963.

[4] U.Oberst, Multidimensional constant linear systems, *Acta Applicandae Mathematicae*, 20:1-175, 1990.

[5] V.P.Palamodov, *Linear Differential Operators with Constant Coefficients*, Springer Verlag, 1970.

[6] H.Pillai and S.Shankar, A behavioural approach to control of distributed systems, *SIAM jl. Control Optim.*, 37:388-408, 1998.

[7] A.Quadrat and D.Robertz, On the blowing-up of stably free behaviours, *preprint*.

[8] S.Shankar, The Nullstellensatz for systems of PDE, *Advances in Applied Mathematics*, 23:360-374, 1999.

[9] S.Shankar, The lattice structure of behaviours, *SIAM jl. Control Optim.*, 39:1817-1832, 2001.

[10] S.Shankar, The evolution of the concept of controllability, *Mathematical and Computer Modelling of Dynamical Systems*, vol.8, 397-406, 2002.

[11] S.Shankar, Geometric completeness of distribution spaces, *Acta Applicandae Mathematicae*, 77:163-180, 2003.

[12] S.Shankar, A Cousin problem for systems of PDE, *Mathematische Nachrichten*, 2007, to appear.

[13] J.C.Willems, Open dynamical systems and their control, *Proceedings of the International Congress of Mathematicians, Berlin*, Vol III: Invited lectures, 697-706, 1998.